

A q -QUEENS PROBLEM

I. GENERAL THEORY

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ABSTRACT. We establish a general counting theory for nonattacking placements of chess pieces with unbounded straight-line moves, such as the queen, and we apply the theory to square boards. We show that the number of ways to place q nonattacking queens on a chessboard of variable size n but fixed shape is a quasipolynomial function of n . The period of the quasipolynomial is bounded by a function of the queen's move directions. Similar conclusions hold for any piece whose moves have unlimited length.

We apply our theory to the square board, to show that the highest-order coefficients of the counting quasipolynomial do not depend on the size of the board. On the other hand, we present simple pieces for which the fourth quasipolynomial coefficient is periodic.

1. INTRODUCTION

The famous n -Queens Problem is to place n nonattacking queens—the largest conceivable number—on an $n \times n$ chessboard, or more broadly, to count the number of placements. (See, for instance, [3, 11].) This problem has no known solution except by individual computation for relatively small values of n . We treat a generalization of the n -Queens Problem, wherein we fix the number of queens, q , and vary n , the size of the board, and the “queen” may be any of a large class of traditional and fairy chess pieces called “riders”. We show (Theorem 3.1) that in each separate problem the number of solutions is a quasipolynomial function of n , which means it is given by a cyclically repeating sequence of polynomials.

The proof is a simple application of the theory of inside-out polytopes [2], which is an extension of Ehrhart's theory of counting lattice points in convex polytopes (cf. [13, Chapter 4]). Our results apply to any pieces with unbounded straight-line moves, such as the queen, rook, or bishop, or the nightrider of fairy chess, which moves arbitrary distances in the directions of a knight's move—in fact, the requirements of our method are the definition of a fairy-chess “rider”; thus, our proof of quasipolynomiality applies to all riders—and only to riders. Our work generalizes in other ways too, as the quasipolynomiality property extends to arbitrary rational convex polygonal shapes and to mixtures of pieces with different moves. For simplicity, though, we assume all pieces have the same moves and we focus mainly on square boards.

One of the key applications of our theory is a framework for finding the coefficients and periods of the highest-degree terms in the counting quasipolynomial. In this paper, aside from a brief survey in Section 5 and a treatment of one-move pieces in Section 4.4, we

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do not go into the details of particular pieces; that is reserved for Part II [5]. There, for pieces that—like the rook, bishop, and queen—move in a subset of the moves of a queen, we establish formulas for the first four quasipolynomial coefficients and prove, by an explicit enumeration of lattice points for each of the relevant subspaces, that the first five quasipolynomial coefficients are constants, independent of n . By contrast, for other kinds of pieces we discuss in Proposition 4.7 how the fourth quasipolynomial coefficient may be non-constant, dependent on n , even for pieces that have only one move, such as the piece that moves arbitrarily far in only one of the directions of a knight’s move.

Now we state the problem more precisely. It has three ingredients: a piece, a board, and a number. The *piece* \mathbb{P} has moves given by integral multiples of a nonempty set \mathbf{M} of non-zero, non-parallel integral vectors $m_r \in \mathbb{R}^2$. A *move* is the difference between a new position and the original position; that is, if a piece is in position $z \in \mathbb{Z}^2$, it may move to any location $z + \kappa m_r$ for $\kappa \in \mathbb{Z}$ and $m_r \in \mathbf{M}$. We call the m_r ’s the *basic moves*. Each one must be reduced to lowest terms; that is, its two coordinates need to be relatively prime, and no basic move may be a scalar multiple of any other. (Thus, the slope of m_r contains all necessary information and can be specified instead of m_r itself.) The *board* consists of the integral points in the interior $t\mathcal{B}^\circ$ of an integral multiple $t\mathcal{B}$ of a rational convex polygon $\mathcal{B} \subset \mathbb{R}^2$ (that is, the vertices of \mathcal{B} have rational coordinates). The number q is the number of pieces that are to occupy places on the board. The rule is that no two pieces may attack each other, or to say it mathematically, if there are pieces at positions z_i and z_j , then $z_j - z_i$ is not a multiple of any m_r .

For instance, the polygon may be the unit square $[0, 1]^2$. The multiple $(n + 1)[0, 1]^2$ has interior points (x, y) for integers $x, y = 1, 2, \dots, n$. That is, for the $n \times n$ square board of the introduction, $t = n + 1$. (A rectangle $[0, a] \times [0, b]$ with rational a and b , whose board is the points of $[(0, ta) \times (0, tb)] \cap \mathbb{Z}^2$, is also covered by our work.) The set \mathbf{M} is $\{(1, 1), (1, -1)\}$ for a bishop, $\{(1, 0), (1, 1), (0, 1), (1, -1)\}$ for a queen, and $\{(2, 1), (1, 2), (2, -1), (1, -2)\}$ for a knight.

This is the place to mention the extensive work of Vaclav Kotěšovec, who collected previous results and produced many new formulas to count non-attacking configurations of chess pieces. His results are reported for instance in his recent book [9] and the related Web site [8]. Kotěšovec’s formulas and numbers were obtained without our theory, so our work, to the extent it duplicates his, is an independent confirmation of his results. More fundamentally, Kotěšovec’s method of work does not rigorously prove the validity of the formulas; our theoretical work therefore complements his calculations by providing, and more fundamentally showing how to provide, proofs.

We took advantage of Kotěšovec’s formulas for bishops and queens to guide some of our investigations. For instance, he found that the quasipolynomial counting formulas tend to have high-degree coefficients that do not vary periodically, that the bishops quasipolynomial appears always to have period 2 when $q > 2$, and that the queens period is computable from Fibonacci numbers. Those observations led us to closely examine the geometry of the problem, which resulted in a proof that the bishops period is indeed not more than 2 and in a confirmation of part of Kotěšovec’s conjecture about the queens period. (See Part II.)

Our results provide a start towards accounting for the periodicity properties of the higher-order coefficients in Kotěšovec’s formulas. The general Ehrhart theory of inside-outside polytopes implies a period that divides the least common multiple of the denominators of

the coordinates of certain points. This least common multiple is called the *denominator* of the problem; it is the denominator D of the inside-out polytope defined in Section 2. In Section 3.2, we provide two methods of approach to understanding something about this denominator. The first approach involves subdeterminants of matrices, but it appears to provide an inefficient bound on the period. The second approach is more geometrical, creating a configuration of points and lines to represent attacks of specified slopes and from which we can calculate the denominator. This second approach allows us to generate sequences of denominators, for increasing numbers of pieces. In particular for queens, it provides coordinate denominators that are Fibonacci numbers (see Part II), a step towards Kotěšovec's conjecture on the period of the q -queens counting quasipolynomial. However, Kotěšovec's conjecture remains open.

In the final section of the paper, we discuss questions related to these ideas and present new research directions. We briefly discuss pieces of different kinds on the same board, pieces on higher-dimensional boards, and even a wild generalization where the attacking moves depend on which piece is attacked as well as which kind does the attacking.

We append a dictionary of notation for the benefit of the authors and readers.

2. TECHNICAL BACKGROUND

The essential tools for our study are hyperplane arrangements and the Ehrhart theory of inside-out polytopes.

In a vector space \mathbb{R}^d , an *arrangement of hyperplanes*, \mathcal{A} , is a finite set of hyperplanes, i.e., linear subspaces of codimension 1. A *region* of a hyperplane arrangement is a connected component of the complement of the union of all the hyperplanes. The *intersection lattice* of \mathcal{A} is the set

$$\mathcal{L}(\mathcal{A}) := \left\{ \bigcap \mathcal{S} : \mathcal{S} \subseteq \mathcal{A} \right\},$$

partially ordered by reverse inclusion. Thus, it is a partially ordered set, it has bottom element $\hat{0} = \mathbb{R}^d$ and top element $\hat{1} = \bigcap \mathcal{A}$; in fact, it is a geometric lattice.

An *inside-out polytope* $(\mathcal{P}, \mathcal{A})$ (see [2], which is the source of the following exposition) is a convex polytope $\mathcal{P} \subseteq \mathbb{R}^d$, which we assume is full-dimensional, together with a hyperplane arrangement \mathcal{A} in \mathbb{R}^d . A *region* of $(\mathcal{P}, \mathcal{A})$ is a nonempty set that is the intersection of \mathcal{P}° , the interior of \mathcal{P} , with a region of the arrangement \mathcal{A} . A *vertex* of $(\mathcal{P}, \mathcal{A})$ is a point obtained as the intersection of $d - k$ linearly independent hyperplanes of \mathcal{A} with a k -dimensional face of \mathcal{P} . (That includes vertices of \mathcal{P} , for which $k = 0$, and any points of intersection of forbidden hyperplanes that lie in the interior of \mathcal{P} , for which $k = d$.) When \mathcal{A} is empty we have just a convex polytope; the vertices are just the vertices of \mathcal{P} . The *intersection semilattice* of $(\mathcal{P}, \mathcal{A})$ is the set

$$\mathcal{L}(\mathcal{P}^\circ, \mathcal{A}) := \{ \mathcal{U} \in \mathcal{L}(\mathcal{A}) : \mathcal{U} \cap \mathcal{P}^\circ \neq \emptyset \},$$

ordered by reverse inclusion. A subspace $\mathcal{U} \in \mathcal{L}(\mathcal{P}^\circ, \mathcal{A})$ is *decomposable* into subspaces \mathcal{U}_1 and \mathcal{U}_2 if $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$ where $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{L}(\mathcal{P}^\circ, \mathcal{A})$, $\text{codim } \mathcal{U} = \text{codim } \mathcal{U}_1 + \text{codim } \mathcal{U}_2$, and for every hyperplane $\mathcal{H} \in \mathcal{L}(\mathcal{P}^\circ, \mathcal{A})$ that contains \mathcal{U} , \mathcal{H} contains either \mathcal{U}_1 or \mathcal{U}_2 .

(In this paper, \mathcal{P} is \mathcal{B}^q , a $2q$ -dimensional polytope that contains all configurations of q pieces in the board, and the hyperplane arrangement is $\mathcal{A}_{\mathbb{P}}$, consisting of hyperplanes that contain all the $2q$ -dimensional points representing configurations of q chess pieces \mathbb{P} in which some pieces attack each other; see the complete definition in Section 3.1.)

A *quasipolynomial* is a function $f(t)$ of positive integers that can be written in the form $e_d(t)t^d + e_{d-1}(t)t^{d-1} + \cdots + e_0(t)$ where each coefficient $e_j(t)$ is a periodic function of t . The least common multiple p of the periods of all the coefficients is the *period* of f . Another way to describe f is as a function that is given by p polynomials, $f_k(t)$ for $k = 0, 1, \dots, p-1$, under the rule $f(t) = f_k(t)$ if $0 < t \equiv k \pmod{p}$. We call the individual polynomials $f_k(t)$ the *constituents* of f . We say f has *degree* d if that is the highest degree of a constituent. (In our quasipolynomials every constituent has the same degree.)

For a positive integer t and a polytope \mathcal{P} , the number of integer points in $t\mathcal{P}^\circ$, or equivalently the number of $(1/t)$ -fractional points in \mathcal{P} , is denoted by $E_{\mathcal{P}}^\circ(t)$. We assume the vertices of \mathcal{P} are rational and let D be the *denominator* of \mathcal{P} , the least common denominator of all their coordinates. Then $E_{\mathcal{P}}^\circ$ is a quasipolynomial function of t , the *open Ehrhart quasipolynomial* of \mathcal{P} . Furthermore, the leading term of every constituent polynomial is $\text{vol}(\mathcal{P})t^d$ where the coefficient is the volume of \mathcal{P} , and the period of this quasipolynomial is a divisor of D ; in particular, if \mathcal{P} has integral vertices, $E_{\mathcal{P}}^\circ$ is a polynomial. (These results are due to Ehrhart; see, e.g., [1].)

An inside-out polytope $(\mathcal{P}, \mathcal{A})$ that has rational vertices has similar properties. Its *open Ehrhart quasipolynomial* is the function $E_{\mathcal{P}, \mathcal{A}}^\circ(t) :=$ the number of $(1/t)$ -fractional points in \mathcal{P}° that do not lie in any of the hyperplanes of \mathcal{A} . The *denominator* $D(\mathcal{P}, \mathcal{A})$ is the least common denominator of the coordinates of all vertices. Given $\mathcal{U} \in \mathcal{L}(\mathcal{A})$, the *volume* $\text{vol}(\mathcal{U} \cap \mathcal{P})$ when $\dim \mathcal{U} < d$ is a relative volume defined in terms of the integral lattice $\mathcal{U} \cap \mathbb{Z}^d$; it is the proportion that the measure of $\mathcal{U} \cap \mathcal{P}$ bears to that of a fundamental domain of $\mathcal{U} \cap \mathbb{Z}^d$. In the case of \mathcal{P} itself, it is the usual volume, since $\mathcal{U} = \mathbb{R}^d$.

Lemma 2.1 ([2, Theorem 4.1]). *The open Ehrhart quasipolynomial has the form*

$$E_{\mathcal{P}, \mathcal{A}}^\circ(t) = e_d t^d + e_{d-1}(t)t^{d-1} + \cdots + e_0(t)t^0,$$

where the coefficient e_d is the volume of \mathcal{P} (a constant) and the coefficients $e_j(t)$ for $j < d$ are periodic functions of t with period that divides the denominator $D(\mathcal{P}, \mathcal{A})$.

A basic formula in the Ehrhart theory of inside-out polytopes [2, Equation (4.4)] is

$$(2.1) \quad E_{\mathcal{P}, \mathcal{A}}^\circ(t) = \sum_{\mathcal{U} \in \mathcal{L}(\mathcal{P}^\circ, \mathcal{A})} \mu(\hat{0}, \mathcal{U}) E_{\mathcal{U} \cap \mathcal{P}^\circ}(t),$$

where μ denotes the Möbius function of $\mathcal{L}(\mathcal{P}^\circ, \mathcal{A})$. This has the following important consequence.

Lemma 2.2. *Suppose $\mathcal{U} \cap \mathcal{P}$ has integral vertices for every $\mathcal{U} \in \mathcal{L}(\mathcal{P}^\circ, \mathcal{A})$ whose codimension is $< k$. Then the coefficients $e_{d-i}(t)$ are constant for all $i \leq k$.*

Proof. Each Ehrhart quasipolynomial on the right-hand side of Equation (2.1) has the form

$$E_{\mathcal{U} \cap \mathcal{P}^\circ}(t) = e_{\dim \mathcal{U}}(\mathcal{U}; t)t^{\dim \mathcal{U}} + e_{\dim \mathcal{U}-1}(\mathcal{U}; t)t^{\dim \mathcal{U}-1} + \cdots + e_0(\mathcal{U}; t)t^0,$$

where each $e_j(\mathcal{U}; t)$ is a periodic function of t and $e_{\dim \mathcal{U}}(\mathcal{U}; t)$ is the $(\dim \mathcal{U})$ -dimensional volume of $\mathcal{U} \cap \mathcal{P}$. If \mathcal{U} has integral vertices the denominator of \mathcal{U} is 1 so each $e_j(\mathcal{U}; t)$ is $e_j(\mathcal{U})$, a constant independent of t . Now,

$$E_{\mathcal{P}, \mathcal{A}}^\circ(t) = \sum_{\mathcal{U} \in \mathcal{L}(\mathcal{P}^\circ, \mathcal{A})} \mu(\hat{0}, \mathcal{U}) \sum_{j=0}^{\dim \mathcal{U}} e_j(\mathcal{U}; t)t^j$$

$$= \sum_{j=0}^d t^j \sum_{\substack{\mathcal{U} \in \mathcal{L}(\mathcal{P}^\circ, \mathcal{A}): \\ \text{codim } \mathcal{U} \leq d-j}} \mu(\hat{0}, \mathcal{U}) e_j(\mathcal{U}; t).$$

Thus,

$$(2.2) \quad e_{d-i}(t) = \sum_{\substack{\mathcal{U} \in \mathcal{L}(\mathcal{P}^\circ, \mathcal{A}): \\ \text{codim } \mathcal{U} \leq i}} \mu(\hat{0}, \mathcal{U}) e_{d-i}(\mathcal{U}; t).$$

If $\text{codim } \mathcal{U} = i$, then $e_{d-i}(\mathcal{U}; t) = \text{vol}(\mathcal{P}^\circ \cap \mathcal{U})$, a constant independent of t . If also $\mathcal{U} \cap \mathcal{P}^\circ$ has integral vertices for all \mathcal{U} with $\text{codim } \mathcal{U} < i$, then all coefficients $e_{d-i}(\mathcal{U}; t) = e_{d-i}(\mathcal{U})$, independent of t , so e_{d-i} is a constant. This is true for all $i \leq k$; thus, all terms t^j with $j \geq d - k$ have constant coefficients $e_j(\mathcal{U})$. \square

Taking $k = 1$ gives a special case of most importance for chess placements.

Lemma 2.3. *If \mathcal{P} has integral vertices, then $e_{d-1}(t)$ is constant.* \square

When \mathcal{U} decomposes into \mathcal{U}_1 and \mathcal{U}_2 , the interval $[\hat{0}, \mathcal{U}]$ has the structure of the product $[\hat{0}, \mathcal{U}_1] \times [\hat{0}, \mathcal{U}_2]$, so that $\mu(\hat{0}, \mathcal{U}) = \mu(\hat{0}, \mathcal{U}_1) \mu(\hat{0}, \mathcal{U}_2)$. This reduction formula is useful, for instance, in the treatment of partial queens in Part II.

3. POLYGONAL BOARDS

3.1. Allowed configurations and the move arrangement.

The secret of the solution is to restate each rule of attack as an equation of a forbidden hyperplane in \mathbb{R}^{2q} . A *labelled configuration* describes the locations of q labelled pieces; it is a point $\mathbf{z} = (z_1, \dots, z_q) \in \mathbb{R}^{2q}$ with each $z_i = (x_i, y_i) \in \mathbb{Z}^2$. The labelled configuration is *allowed* or *forbidden* depending on whether or not it violates every attack equation. An *attack equation* is a linear constraint on \mathbf{z} expressing the fact that labelled pieces \mathbb{P}_i and \mathbb{P}_j attack each other; in mathematical terms, that $z_j - z_i$ is a multiple of a move m_r .

To express an attack in the configuration space \mathbb{R}^{2q} , observe that $z_j - z_i \in \langle m_r \rangle$ can be rewritten as $(z_j - z_i) \perp m_r^\perp$, or, $(z_j - z_i) \cdot m_r^\perp = 0$, where m_r^\perp denotes any nonzero vector orthogonal to m_r . The equation $(z_j - z_i) \cdot m_r^\perp = 0$ is the equation of a hyperplane in the configuration space (the *move hyperplane* associated to the move m_r) whose points are forbidden labelled configurations. These forbidden hyperplanes in the configuration space form an arrangement of hyperplanes, $\mathcal{A}_{\mathbb{P}}$, which we call the *move arrangement* of \mathbb{P} . There are $\binom{q}{2} |\mathbf{M}|$ of these hyperplanes.

For specificity, for each basic move vector $m_r = (c_r, d_r)$, we define $m_r^\perp := (d_r, -c_r)$, which is m_r rotated 90° counterclockwise; thus, m_r^\perp points to the left side of the move line.

We did not mention the board in these definitions. Any integral point in configuration space \mathbb{R}^{2q} lies in some inflated board $t\mathcal{B}^q$. However, the fact that the board can engulf any integral point makes our polytopal approach awkward. Therefore, we often reduce the integral configuration $\mathbf{z} \in t\mathcal{B}^q$ to a fractional configuration $\mathbf{z}' = t^{-1}\mathbf{z} \in \mathcal{B}^q$. The denominators of the components of \mathbf{z}' tell us which dilates $t\mathcal{B}^q$ contain a corresponding integral point $t\mathbf{z}'$, since $t\mathbf{z}'$ is integral precisely when t is a multiple of the least common denominator of the components of \mathbf{z}' . We refer to either $\mathbf{z} \in t\mathcal{B}^q \cap \mathbb{Z}^{2q}$ or $\mathbf{z} \in \mathcal{B}^q \cap \mathbb{Q}^{2q}$ as a *configuration*, assuming that the context will make clear whether we mean an integral or fractional configuration.

3.2. Counting configurations.

An *unlabelled configuration* is the multiset of planar points z_1, \dots, z_q ; it corresponds to having unlabelled pieces. Unlabelled configurations are what we really want to count. Since a point with any $z_j = z_i$ is forbidden, the same number $q!$ of allowed unlabelled configurations corresponds to each allowed labelled configuration.

Let $u_{\mathbb{P}}(t)$ be the number of allowed unlabelled configurations and $o_{\mathbb{P}}(t)$ the number of allowed labelled configurations with all pieces in the interior of the dilated board $t\mathcal{B}$, where t is a positive integer; thus $u_{\mathbb{P}}(t) = o_{\mathbb{P}}(t)/q!$. Let $|\mathcal{B}|$ denote the area of \mathcal{B} .

Theorem 3.1. *The number $u_{\mathbb{P}}(t)$ of permitted unlabelled configurations of q pieces in $t\mathcal{B}$ is given by a quasipolynomial function of t , of degree $2q$ with leading coefficient $|\mathcal{B}|^q/q!$.*

Proof. We prove the theorem by showing that $o_{\mathbb{P}}(t)$ is a quasipolynomial with suitable properties.

We already know that $o_{\mathbb{P}}(t)$ is the number of integral lattice points in the interior of $t\mathcal{B}^q$ that are not in any of the forbidden hyperplanes. Since the hyperplanes are homogeneous, the integral lattice points in $t\mathcal{B}^q$ can be scaled to t^{-1} -fractional points in \mathcal{B}^q . Technically, inside-out theory applies to the count of these fractional points in \mathcal{B}^q . The theory also requires that the forbidden hyperplanes have rational equations, which they do. According to the theory, $o_{\mathbb{P}}(t)$ is a quasipolynomial function of t whose degree is $\dim \mathcal{B}^q$, which is $2q$, and whose leading coefficient is the volume of \mathcal{B}^q . This volume is obviously $|\mathcal{B}|^q$. \square

3.2.1. Bounds on the period.

This theorem says nothing about the period. We want to bound the period by deriving the denominator from the plane geometry of \mathcal{B} (which gives the boundary inequalities) and from \mathbf{M} (which gives the attack constraints).

Let the boundary inequalities (with integral coefficients) of the polygon \mathcal{B} be $a_jx + b_jy \leq \beta_j$ for $1 \leq j \leq \omega$. The pieces have coordinate (column) vectors z_1, z_2, \dots, z_q , which must satisfy $a_jx_i + b_jy_i \leq \beta_j$ for all $1 \leq i \leq q$ and $1 \leq j \leq \omega$. Then the system $A\mathbf{z} = \mathbf{b}$ in Equation (3.1) contains all the equations that determine any one vertex of the inside-out polytope $(\mathcal{B}^q, \mathcal{A}_{\mathbb{P}})$.

$$(3.1) \quad \begin{pmatrix} M & -M & 0 & 0 & \cdots & 0 & 0 \\ M & 0 & -M & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ M & 0 & 0 & 0 & \cdots & 0 & -M \\ 0 & M & -M & 0 & \cdots & 0 & 0 \\ 0 & M & 0 & -M & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & M & 0 & 0 & \cdots & 0 & -M \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & M & -M \\ \hline B & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & B & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & B \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \hline \beta \\ \beta \\ \vdots \\ \beta \end{pmatrix},$$

where M and B are the matrices

$$M := \begin{pmatrix} m_1^\perp \\ m_2^\perp \\ \vdots \\ m_{|\mathbf{M}|}^\perp \end{pmatrix} \quad B := \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_\omega & b_\omega \end{pmatrix},$$

containing the row vectors m_r^\perp , and where β is the column vector of constant terms $\beta_1, \dots, \beta_\omega$. We define A' to be the top half of A .

A fundamental fact from linear algebra is the following lemma.

Lemma 3.2. *The coordinates $z_i = (x_i, y_i)$ belong to a vertex of the inside-out polytope if and only if there are k attack equations and $2q - k$ boundary equations that uniquely determine those coordinates.*

A vertex corresponds to a set of violated boundary and attack constraints that determines uniquely (up to translation) a particular placement of q labelled pieces on the lattice points contained in some integer dilate $t\mathcal{B}$ of the closed board.

3.2.2. Cramer's rule and rectangular boards. Alternatively, we might investigate the system in Equation (3.1) directly as a matrix. It follows by Cramer's Rule and Lemma 3.2 that every denominator of an inside-out vertex divides a $2q \times 2q$ subdeterminant of A . The period of the counting quasipolynomial of $(\mathcal{B}^q, \mathcal{A}_{\mathbb{P}})$ is a divisor of the least common multiple of all $2q \times 2q$ subdeterminants of A . This quantity is not so easy to determine, but for rectangular boards there is a way to estimate it, if there are not too many moves.

For a rational rectangular board with sides on the axes, say $\mathcal{B} = [0, a] \times [0, b]$, the dilation $(n+1)\mathcal{B}^\circ$ contains integral points in an $(na-1) \times (nb-1)$ rectangle. (This is not precisely what one wants of a rectangular board; the proportions should remain fixed under dilation. However, it is what our method handles.) The augmented matrix is

$$(B \quad \beta) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & a \\ 0 & 1 & b \end{pmatrix}.$$

Rearranging the bottom half of A in (3.1), it takes the form $\begin{pmatrix} -I_{2q} \\ I_{2q} \end{pmatrix}$. Consequently, for a rectangular board the values of the $2q \times 2q$ subdeterminants of A are the values of the subdeterminants of any order of A' , the top half of A . Thus, the period of the counting quasipolynomial divides $\text{lcmd}(A')$, the least common multiple of all subdeterminants of A' . The value of $\text{lcmd}(A')$ is the only general theoretical bound we know for the period without finding the denominator itself.

In general $\text{lcmd}(A')$ is difficult to compute. Two of us studied it and found a computable formula that applies as long as the moves matrix M has up to two rows [6]. Observe that A' is the Kronecker product $\mathbf{H}_q^T \otimes M$ where \mathbf{H}_q^T is the matrix consisting of one row for each pair of different pieces, say i and j , in which all columns are zero except for a 1 in the column of piece i and a -1 in that of piece j . Thus, \mathbf{H}_q is the oriented incidence matrix of the complete graph K_q , which is well known to be totally unimodular with rank $q-1$. In this situation we can calculate $\text{lcmd}(A')$ when M has two rows by means of [6, Corollary 2], which in terms

of $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ and H_q^T states that, for a piece with two moves,

$$(3.2) \quad \text{lcmd}(H_q^T \otimes M) = \text{lcm}((\text{lcmd } M)^{q-1}, \text{LCM}_{p=1}^{\lfloor q/2 \rfloor} ((m_{11}m_{22})^p - (m_{12}m_{21})^p)^{\lfloor q/2p \rfloor}),$$

where LCM_p denotes the least common multiple of all the terms for p in the indicated range. We conclude that, for a piece with one or two moves on a rectangular board with sides parallel to the axes, the period of $o_{\mathbb{P}}(t)$ is a divisor of the right-hand side of Equation (3.2). This applies to the bishop, where the right-hand side equals 2^q (see [6]).

Unfortunately, this bound on the period is far from sharp (see Tables 5.1, 5.2, and 5.3) and, worse, the theory of [6] does not apply to a matrix M with more than two rows, which means a piece with more than two move directions. For such matrices, e.g., for the queen and nightrider, we have to calculate the determinantal upper bound $\text{lcmd}(A')$ on the period separately for each value of q . One hopes that Equation (3.2) can be generalized to $m \times 2$ matrices.

3.3. Counting configuration types.

For each basic move $m_r \in \mathbf{M}$ from a fixed location, the line $\langle m_r \rangle$ is naturally directed, so it has a left and right side. Given a nonattacking configuration of q pieces, record, for each piece \mathbb{P}_i at location z_i and each move line $z_i + \langle m_r \rangle$ through \mathbb{P}_i , oriented in the direction of m_r , the indices of the points \mathbb{P}_j that lie on the left side of the line. The set of these lists, for every pair, is the *combinatorial type* of the labelled configuration, briefly the *labelled configuration type*.

Another way to describe a labelled configuration type is by building a nonattacking configuration, one piece at a time. Place the first piece, labelled \mathbb{P}_1 , at $z_1 \in t\mathcal{B}^\circ \cap \mathbb{Z}^2$. This creates $|\mathbf{M}|$ lines through z_1 that may not be occupied by any other piece, and $2|\mathbf{M}|$ regions that may be occupied. The second piece, \mathbb{P}_2 at z_2 , will be in one of these regions. Now we have $|\mathbf{M}|$ forbidden lines through z_2 ; these lines in combination create permitted regions that will be occupied by the other pieces. Placing the third piece further subdivides these into a larger number of regions, and similarly with each placement up to the last piece. The sequence of choices of region is equivalent to the labelled configuration type.

By forgetting the order of the pieces we have an *unlabelled combinatorial type* of configuration, for short an *unlabelled configuration type*. We want to know how many unlabelled configuration types there are.

Lemma 3.3. *There are $q!$ labelled combinatorial configuration types for each unlabelled type. Thus, $o_{\mathbb{P}}(t) = q!u_{\mathbb{P}}(t)$.*

Proof. In the left-to-right direction $-m_r^\perp$ perpendicular to a move line $\langle m_r \rangle$, the q labelled pieces appear in a definite order, $(\mathbb{P}_1, \mathbb{P}_2, \dots, \mathbb{P}_q)$. This is indicated by the left-side list of the i th piece with respect to m_r , which is $\{1, \dots, i-1\}$. Renumbering the pieces changes the order, hence the left-side list of at least one piece, and therefore the labelled configuration type. \square

Lemma 3.4. *The labelled combinatorial configuration types are in one-to-one correspondence with the regions of $(\mathcal{B}^q, \mathcal{A}_{\mathbb{P}})$.*

Proof. We need to be more formal about labelled configuration types. A configuration on the t -fold board $t\mathcal{B}$, in terms of coordinates, is $\mathbf{z} = (z_1, z_2, \dots, z_q) \in (t\mathcal{B}^\circ)^q \cap \mathbb{Z}^{2q}$. We normalize

this to a fractional configuration $t^{-1}\mathbf{z} \in (\mathcal{B}^\circ)^q$. The normalized configurations are just as good as the integral ones as far as describing configuration types, because the inequalities that describe types are homogeneous.

The type of a configuration \mathbf{z} is a list of lists of lists. The i th point has lists $L_{ir} = \{j : \mathbb{P}_j \text{ is on the left side of the } r\text{th move line through } \mathbb{P}_i\}$, one for each basic move m_r . That is,

$$L_{ir} = \{j : (z_j - z_i) \cdot m_r^\perp > 0\}.$$

This inequality is precisely what defines the positive halfspace of a hyperplane in $\mathcal{A}_\mathbb{P}$; the collection of all such inequalities derived from the configuration \mathbf{z} determines a subset of the interior of \mathcal{B}^q , which is nonvoid because it contains the fractional configuration $t^{-1}\mathbf{z}$. Consequently, a configuration determines a region of $(\mathcal{B}^q, \mathcal{A}_\mathbb{P})$.

Conversely, any region contains a fractional point $t^{-1}\mathbf{z} \in t^{-1}\mathbb{Z}^{2q}$ for a sufficiently large integer t . Therefore, it corresponds to one or more configuration types. However, it cannot correspond to more than one configuration type, because the inequalities that define the region determine which indices j are in which list L_{ir} for each i and r . \square

Recall that $u_\mathbb{P}(t)$ and $o_\mathbb{P}(t)$ are quasipolynomials, whose constituent polynomials are $u_{\mathbb{P},0}(t), \dots, u_{\mathbb{P},p-1}(t)$ and $o_{\mathbb{P},0}(t), \dots, o_{\mathbb{P},p-1}(t)$, respectively.

Theorem 3.5. *The number of unlabelled combinatorial types of configuration equals $|u_{\mathbb{P},0}(0)|$, the constant term of the 0th constituent of $u_\mathbb{P}(t)$. The number of labelled configuration types equals $|o_{\mathbb{P},0}(0)|$.*

Proof. By [2, Theorem 4.1], the number of regions of $(\mathcal{B}^q, \mathcal{A}_\mathbb{P})$ is $|o_{\mathbb{P},0}(0)|$. \square

Still another way to look at the type of a configuration is through isotopy. Two labelled configurations \mathbf{z} and \mathbf{z}' are *isotopic* if one can be deformed into the other by a continuous movement in the configuration space \mathbb{R}^{2q} without at any time crossing a forbidden hyperplane. It is clear from the correspondence between lists and regions in the proof of Lemma 3.4, and the convexity of regions, that this is possible if and only if \mathbf{z} and \mathbf{z}' have the same left-side lists, and then the isotopism can be performed along a line segment in the interior of $t\mathcal{B}^q$.

On the other hand, one might ask about discrete isotopy, where we move one piece at a time on the board. A *discrete isotopism* is a sequence of steps, $\mathbf{z} = \mathbf{z}^0 \rightarrow \mathbf{z}^1 \rightarrow \dots \rightarrow \mathbf{z}^k = \mathbf{z}'$ where \mathbf{z}^{j-1} and \mathbf{z}^j differ only by making a legitimate move of a single piece that does not change the combinatorial type of the configuration. One should allow any amount of inflation, i.e., one should be allowed to multiply all the coordinates by a very large positive integer before performing the isotopism (which one can think of as replacing the $(1/t)$ -lattice by a finer $(1/kt)$ -lattice). One would naively expect this to be equivalent to continuous isotopy, and indeed it is, once we overcome two difficulties.

First, if there is only one basic move m , the only configurations that can be reached from an allowed configuration \mathbf{z} by allowed moves are those in the line through \mathbf{z} in the direction of m (in some z_i plane). Therefore, there must be more than one move in \mathbf{M} .

Second, even if there are two basic moves, there can be configurations that are unreachable from each other in a board $t\mathcal{B}$ if t is fixed; for instance, in a configuration of bishops no moves can change the numbers of bishops on squares of each color. That problem is solvable by inflation.

Theorem 3.6. *If $|\mathbf{M}| \geq 2$, isotopy and discrete isotopy produce the same equivalence relation on allowed configurations.*

The proof begins with a planar lemma. Note that any two basic moves are nonparallel.

Lemma 3.7. *Given two basic moves, m_1 and m_2 , there is a sequence of moves that takes a piece from $z = (x, y) \in \mathbb{Z}^2$ to $z' = (x', y') \in \mathbb{Z}^2$ if and only if $(x' - x, y' - y)$ is divisible by $\det(m_1, m_2)$.*

Proof. There exists such a sequence if and only if there is an integral solution to $\kappa m_1 + \lambda m_2 = (x' - x, y' - y)$. Let $C = \begin{pmatrix} m_1 & m_2 \end{pmatrix}$. The equation to be solved is $\begin{pmatrix} \kappa \\ \lambda \end{pmatrix} C = \begin{pmatrix} x' - x \\ y' - y \end{pmatrix}$. Inverting,

$$\begin{pmatrix} \kappa \\ \lambda \end{pmatrix} = (\det C)^{-1} C^* \begin{pmatrix} x' - x \\ y' - y \end{pmatrix},$$

where C^* is the cofactor matrix, which is an integral matrix. By the assumptions on moves, the greatest common divisor of the entries in either column of C is 1; thus, an integral solution exists if and only if $x' - x$ and $y' - y$ are multiples of $\det C$. \square

Proof of Theorem 3.6. Choose two basic moves $m_1, m_2 \in \mathbf{M}$.

Let $\mathbf{z} = (z_1, \dots, z_q)$, $\mathbf{z}' = (z'_1, \dots, z'_q) \in \mathbb{Z}^{2q}$ be nonattacking configurations of the same combinatorial type on the dilated board $t\mathcal{B}^\circ$, where $t > 0$. No restriction hyperplane separates them; that is, they lie in the same open region R of $\mathcal{A}_\mathbb{P}$. We want a discrete isotopism from \mathbf{z} to \mathbf{z}' , that is, is a sequence of moves of individual pieces that gives a sequence of (fractional) configurations lying in $R \cap \mathcal{B}^\circ \cap \tau^{-1}\mathbb{Z}^{2q}$ for some $\tau \in \mathbb{Z}_{>0}$.

A sequence of individual moves is expressed (disregarding its order) by solving the Diophantine equation

$$\kappa_1 \begin{pmatrix} m_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \dots + \kappa_q \begin{pmatrix} 0 \\ \vdots \\ 0 \\ m_1 \end{pmatrix} + \lambda_1 \begin{pmatrix} m_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \dots + \lambda_q \begin{pmatrix} 0 \\ \vdots \\ 0 \\ m_2 \end{pmatrix} = \begin{pmatrix} \mathbf{z}'_1 - \mathbf{z}_1 \\ \mathbf{z}'_2 - \mathbf{z}_2 \\ \vdots \\ \mathbf{z}'_q - \mathbf{z}_q \end{pmatrix}.$$

This is a set of q independent equations:

$$\kappa_i m_1 + \lambda_i m_2 = z'_i - z_i$$

for $i = 1, \dots, q$. We know from Lemma 3.7 when they are solvable: if and only if $\det C$ divides every component of $\mathbf{z}' - \mathbf{z}$. If we multiply the entire board by k , replacing $\mathbf{z}' - \mathbf{z}$ by $k(\mathbf{z}' - \mathbf{z})$, this condition is satisfied; thus, using kt as the board dilation factor, we get a walk $k\mathbf{I} := (k\mathbf{z} = \mathbf{z}^0, \mathbf{z}^1, \dots, \mathbf{z}^l = k\mathbf{z}')$, from $k\mathbf{z}$ to $k\mathbf{z}'$ in $(kt\mathcal{B}^\circ)^q \cap \mathbb{Z}^{2q}$ consisting of integral multiples of moves m_1 and m_2 . We normalize $k\mathbf{I}$ to lie in $(t\mathcal{B}^\circ)^q \cap k^{-1}\mathbb{Z}^{2q}$ through division by k ; this gives a walk

$$\mathbf{I} = (\mathbf{z} = k^{-1}\mathbf{z}^0, k^{-1}\mathbf{z}^1, k^{-1} \dots, k^{-1}\mathbf{z}^l = \mathbf{z}')$$

in $(t\mathcal{B}^\circ)^q \cap k^{-1}\mathbb{Z}^{2q}$ where each $k^{-1}\mathbf{z}^i - k^{-1}\mathbf{z}^{i-1} \in k^{-1}\mathbb{Z}^{2q}$ is a move of a single piece by an integral multiple of $k^{-1}m_1$ or $k^{-1}m_2$.

Define $\mathbf{I}_0 := (0, k^{-1}\mathbf{z}^1 - \mathbf{z}, k^{-1}\mathbf{z}^2 - \mathbf{z}, \dots, \mathbf{z}' - \mathbf{z})$, the walk \mathbf{I} translated to the origin; thus we may write $\mathbf{I} = \mathbf{z} + \mathbf{I}_0$.

We squeeze \mathbf{I} into R by shrinking and replicating it. The line segment $[\mathbf{z}, \mathbf{z}']$ lies in $R \cap (t\mathcal{B}^\circ)^q$ by the convexity of regions and of the polytope $(t\mathcal{B}^\circ)^q$. For some $\delta \in \mathbb{R}_{>0}$ the segment has a δ -neighborhood U contained in $R \cap (t\mathcal{B}^\circ)^q$. By taking a sufficiently large divisor $\tau \in \mathbb{Z}_{>0}$ we can ensure that $\mathbf{z}'' + \tau^{-1}\mathbf{I}_0$ is contained in U for every $\mathbf{z}'' \in [\mathbf{z}, \mathbf{z}' - \tau^{-1}(\mathbf{z}' - \mathbf{z})]$.

In particular, that is true for every $\mathbf{z}'' = \mathbf{z} + (j - 1)\tau^{-1}(\mathbf{z}' - \mathbf{z})$ such that $j \in \{1, \dots, \tau\}$. Consequently, the concatenated sequence

$$\mathbf{z} + \tau^{-1}\mathbf{I}_0, \mathbf{z} + 1\tau^{-1}(\mathbf{z}' - \mathbf{z}) + \tau^{-1}\mathbf{I}_0, \dots, \mathbf{z} + (\tau - 1)\tau^{-1}(\mathbf{z}' - \mathbf{z}) + \tau^{-1}\mathbf{I}_0$$

is a walk from \mathbf{z} to \mathbf{z}' in $(t\mathcal{B}^\circ)^q \cap (k\tau)^{-1}\mathbb{Z}^{2q}$ by $(1/k\tau)$ -fractions of legal moves, is contained in R , and is therefore a discrete isotopism from \mathbf{z} to \mathbf{z}' . \square

4. THE SQUARE BOARD

From here on we investigate properties that apply to a piece on the $n \times n$ square board. This board is the simplest one and has the best properties; most especially, on the square board it is easiest to get information about the counting quasipolynomials. The board consists of the integral points in the dilate $(n + 1)\mathcal{B}^\circ = (n + 1)(0, 1)^2$; the associated polytope is $\mathcal{P} = \mathcal{B}^q = [0, 1]^{2q}$ with $\mathcal{P}^\circ = \mathcal{B}^{\circ q} = (0, 1)^{2q}$. (We assume that $q > 0$.) We adopt the short notation $[n] := \{1, \dots, n\}$ so that the set of points of the integral board is

$$[n]^2 = (n + 1)(0, 1)^2 \cap \mathbb{Z}^2.$$

4.1. Theory of the square board.

For a piece \mathbb{P} the inside-out polytope is $([0, 1]^{2q}, \mathcal{A}_{\mathbb{P}})$. Write $u_{\mathbb{P}}(q; n)$ for the number of nonattacking positions of q indistinguishable pieces \mathbb{P} on the $n \times n$ board, and $o_{\mathbb{P}}(q; n)$ for the number of such positions of q distinguishable pieces \mathbb{P} ; then

$$u_{\mathbb{P}}(q; n) = \frac{1}{q!} o_{\mathbb{P}}(q; n) = \frac{1}{q!} E_{\mathcal{P}^\circ, \mathcal{A}_{\mathbb{P}}}(n + 1) = \frac{1}{q!} E_{(0, 1)^{2q}, \mathcal{A}_{\mathbb{P}}}(n + 1).$$

This is a quasipolynomial function of $t = n + 1$, which we expand as

$$u_{\mathbb{P}}(q; n) = \gamma_{2q}(n)n^{2q} + \gamma_{2q-1}(n)n^{2q-1} + \gamma_{2q-2}(n)n^{2q-2} + \dots + \gamma_0(n)n^0.$$

Its leading coefficient is a constant, $\gamma_{2q}(n) = \gamma_{2q}$ (since it is a disguised Ehrhart quasipolynomial) and its period is a divisor of the denominator $D([0, 1]^{2q}, \mathcal{A}_{\mathbb{P}})$. As $n = t - 1$, the number of unlabelled combinatorial types of configuration equals $|u_{\mathbb{P}}(q; -1)|$.

A typical approach to finding $u_{\mathbb{P}}(q; n)$ is to evaluate it at enough small values of n by counting the non-attacking configurations, finding the period p in some way (preferably by bounding the denominator $D(\mathcal{P}, \mathcal{A}_{\mathbb{P}})$), and using that information to interpolate the coefficients of the p constituent polynomials. If only the volume of \mathcal{P} is known, $2pq$ values of n are needed. Any preliminary information about $u_{\mathbb{P}}(q; n)$ can reduce that number. For example, Theorem 4.5 reduces the number of needed values by $2p - 1$ by giving a formula for the second coefficient γ_{2q-1} and proving constancy of γ_{2q-2} . The following property reduces it by two more. Here δ_{ij} is the Kronecker delta.

Proposition 4.1. *On the square board, for any piece \mathbb{P} , $u_{\mathbb{P}}(q; 0) = 0$ and $u_{\mathbb{P}}(q; 1) = \delta_{q1}$.*

Proof. Clearly, there are no ways to place $q > 0$ unlabelled pieces on the 0×0 board and δ_{q1} ways to place them on the 1×1 board, the latter because two pieces attack when they are placed on the same spot. \square

4.2. One or two pieces.

Now we examine exceedingly small numbers of pieces, i.e., $q = 1$ and 2 .

A simple observation is that $u_{\mathbb{P}}(1; n) = n^2$ for any piece. Setting $n = -1$, there is (of course) one combinatorial type.

There is a (relatively) simple way to calculate $u_{\mathbb{P}}(2; n)$. Define $\alpha_{\mathbb{P}}(n)$ to be the number of attacking configurations of two labelled pieces \mathbb{P} (which may occupy the same position; that is considered attacking). Then

$$(4.1) \quad u_{\mathbb{P}}(2; n) = \frac{1}{2!} o_{\mathbb{P}}(2; n) = \frac{1}{2} [n^4 - \alpha_{\mathbb{P}}(n)].$$

Finding $\alpha_{\mathbb{P}}(n)$ is easy in principle although nontrivial in detail. (See Equation (4.6).) To begin with, consider a move $(c, d) \in \mathbf{M}$, whose slope is the rational fraction d/c , and let

$$l^{d/c}(b) := \text{the line in } \mathbb{R}^2 \text{ with slope } d/c \text{ and } y\text{-intercept } b.$$

We allow $d/c = 1/0 = \infty$, in which case b is instead the x -intercept. Define

$$l_{\mathbb{B}}^{d/c}(b) := l^{d/c}(b) \cap [n]^2,$$

the set of positions on the $n \times n$ board $[n]^2$ that lie on the line $l^{d/c}(b)$. The multiset

$$\mathbf{L}^{d/c}(n) := \{|l_{\mathbb{B}}^{d/c}(b)| : l_{\mathbb{B}}^{d/c}(b) \neq \emptyset\}$$

is finite and the sum of its entries is n^2 . We need to know the exact contents of $\mathbf{L}^{d/c}(n)$. Two cases are elementary:

$$\mathbf{L}^{0/1}(n) = \mathbf{L}^{1/0}(n) = \{n^n\},$$

$$\mathbf{L}^{1/1}(n) = \mathbf{L}^{-1/1}(n) = \{1^2, 2^2, \dots, (n-1)^2, n^1\}.$$

Lemma 4.2. *Assume $0 < c \leq d$ are relatively prime integers. Let $c' := n \bmod c$ and $d' := n \bmod d$. The multiplicities of line sizes in $\mathbf{L}^{d/c}(n)$ are as in the following table:*

Line size	$1 \leq l < \lfloor \frac{n}{d} \rfloor$	$\lfloor \frac{n}{d} \rfloor$	$\lfloor \frac{n}{d} \rfloor + 1$
Multiplicity	$2cd$	$(d - d')(n - c \lfloor \frac{n}{d} \rfloor) + c(d' + d)$	$d'(n - c \lfloor \frac{n}{d} \rfloor)$

Proof. Let $\delta := \lfloor n/d \rfloor = (n - d')/d$; note that $\delta \leq \lfloor n/c \rfloor = (n - c')/c$.

Each nonempty line $l_{\mathbb{B}}^{d/c}(b)$ has a lowest point

$$(x, y) \in Z := \{(x, y) \in [n]^2 : x \leq c \text{ or } y \leq d\},$$

and conversely, each point in Z is the lowest point of a different line $l_{\mathbb{B}}^{d/c}(b)$. If we rename the line $l_{\mathbb{B}}(x, y)$, the naming is unique and the points on $l_{\mathbb{B}}(x, y)$ are the points of the form $(x, y) + k(c, d)$ for $k = 0, \dots, \bar{k}$, where \bar{k} is the largest integer such that $(x, y) + \bar{k}(c, d) \in [n]^2$. Solving this last restriction for \bar{k} , we find that

$$x \leq n \implies \bar{k} \leq (n - x)/c \text{ and } y \leq n \implies \bar{k} \leq (n - y)/d.$$

Hence, $\bar{k} = \min(\lfloor (n - x)/c \rfloor, \lfloor (n - y)/d \rfloor)$.

In order to calculate the cardinality of a line $l_{\mathbb{B}}(x, y)$ for $(x, y) \in Z$ we pick out special subrectangles in $[n]^2$ (illustrated in Figure 4.1). First are the lower and left borders:

$$I := \{(x, y) \in [n]^2 : y \leq d\}, \quad J := \{(x, y) \in [n]^2 : x \leq c\}.$$

Define new $c \times d$ rectangles on the bottom edge, from right to left,

$$I_i := \{(x, y) \in I : n - ci < x \leq n - c(i - 1)\} \quad \text{for } i = 1, \dots, \delta,$$

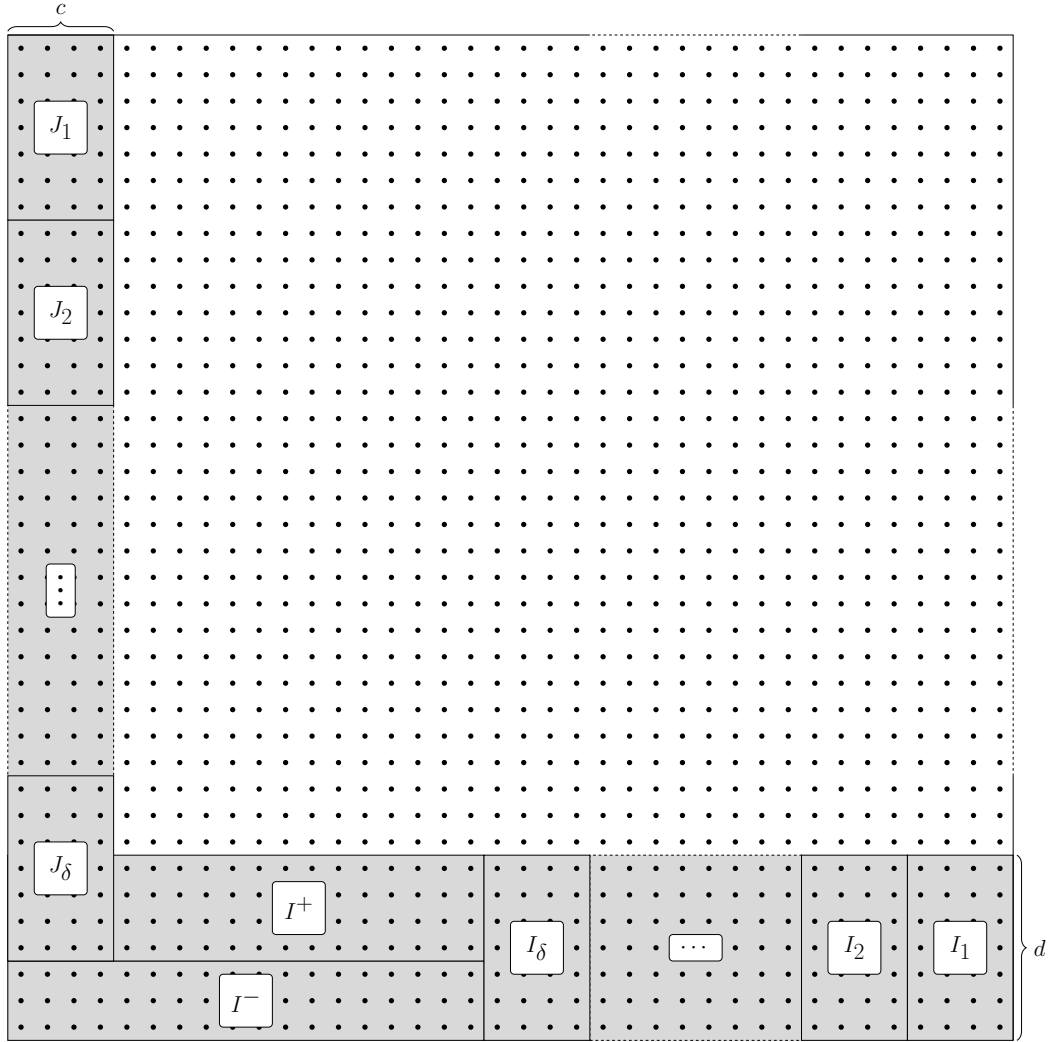


FIGURE 4.1. The division of $[n]^2$ into the shaded border region Z , with its subdividing rectangles, and the remainder of the board. (The illustration shows the case where I_δ and J_δ do not overlap.)

and on the left edge, from the top down,

$$J_j := \{(x, y) \in J : n - dj < y \leq n - d(j - 1)\} \quad \text{for } j = 1, \dots, \delta.$$

Thus, I_1 occupies the bottom right corner of $[n]^2$ and J_1 occupies the top left corner of $[n]^2$. Also, J_δ occupies the upper half of the left end of I .

Then, subdivide the remainder of Z (that is, the part of I to the left of I_δ and not in J_δ) into lower and upper halves:

$$\begin{aligned} I^- &:= \{(x, y) \in I : y \leq d', \ 1 \leq x \leq n - c\delta\}, \\ I^+ &:= \{(x, y) \in I : y > d', \ c + 1 \leq x \leq n - c\delta\}. \end{aligned}$$

There is a critical value of x , namely, $n - c\delta$, such that $\bar{k} = \lfloor (n - x)/c \rfloor$ if $x > n - c\delta$ and $\bar{k} = \lfloor (n - y)/d \rfloor$ if $x \leq n - c\delta$. Hence, we know the size of any line by the formula

$$(4.2) \quad |l_{\mathcal{B}}(x, y)| = \bar{k} + 1 = \begin{cases} \delta + 1, & \text{if } (x, y) \in I^-, \\ \delta, & \text{if } (x, y) \in I^+, \\ i, & \text{if } (x, y) \in I_i \text{ for } i \leq \delta, \\ j, & \text{if } (x, y) \in J_j \text{ for } j \leq \delta. \end{cases}$$

From Equation (4.2) we can write down the multiplicities of all line sizes in the multiset $\mathbf{L}^{d/c}(n)$ by counting the base points (x, y) in each case. We obtain the multiplicities stated in the lemma.

The rectangles I_δ and J_δ do not overlap if and only if the left end of I_δ , at $x = n - c\delta + 1$, is to the right of the right edge of J_δ , at $x = c$; that is, if and only if $n - (c + 1)\delta \geq 0$; equivalently, $\delta = \lfloor n/c \rfloor$. As the width of I^+ is exactly $(n - c\delta) - c$, if there is overlap then I^+ is the overlap and has negative width, so our computation subtracts exactly the amount necessary to correct for double counting of the lines based at $(x, y) \in I_\delta \cap J_\delta$. (In this case $I^+ := \{(x, y) \in I : y > d', c \geq x > n - c\delta\}$.) Thus, our formula works whether or not overlap occurs. \square

Now, define $\alpha^{d/c}(n)$ to be the number of ordered pairs of positions that attack each other along slope d/c . Thus,

$$\alpha^{d/c}(n) := E_{(0,1)^4 \cap \mathcal{H}_{12}^{d/c}}(n + 1),$$

the open Ehrhart quasipolynomial of the subpolytope $[0, 1]^4 \cap \mathcal{H}_{12}^{d/c}$ of $[0, 1]^4$ that satisfies the equation of attack, $(z_2 - z_1) \cdot (c, d)^\perp = 0$ (we write $\mathcal{H}_{12}^{d/c}$ for the subspace defined by this equation). Counting attacking pairs of positions shows that

$$\alpha^{d/c}(n) := \sum_{l \in \mathbf{L}^{d/c}(n)} l^2.$$

The subpolytope is 3-dimensional so the degree of $\alpha^{d/c}(n)$ is 3; therefore its leading coefficient, which is the (relative) volume of $[0, 1]^4 \cap \mathcal{H}_{12}^{d/c}$, is

$$\bar{\alpha}^{d/c} := \lim_{n \rightarrow \infty} \frac{\alpha^{d/c}(n)}{n^3}.$$

The values of $\alpha^{d/c}(n)$ and $\bar{\alpha}^{d/c}$ have to be computed separately for each slope. Two important yet easy examples are

$$(4.3) \quad \begin{aligned} \alpha^{0/1}(n) &= \alpha^{1/0}(n) = n^3, & \bar{\alpha}^{0/1} &= \bar{\alpha}^{1/0} = 1; \\ \alpha^{\pm 1/1}(n) &= \sum_{i=1}^n i^2 + \sum_{i=1}^{n-1} i^2 = \frac{n(2n^2 + 1)}{3}, & \bar{\alpha}^{\pm 1/1} &= \frac{2}{3}. \end{aligned}$$

There are general formulas for when $c, d \neq 0$.

Proposition 4.3. *For relatively prime positive integers c and d with $c \leq d$, let $c' := n \bmod c$ and $d' := n \bmod d$. The number of ordered pairs of positions that attack each other along*

lines of slope d/c is

$$(4.4) \quad \begin{aligned} \alpha^{d/c}(n) &= E_{(0,1)^4 \cap \mathcal{H}_{12}^{d/c}}(n+1) \\ &= \left\{ \left(\frac{1}{d} - \frac{c}{3d^2} \right) n^3 + \frac{c}{3} n \right\} + \frac{d'(d-d')}{d^2} \left\{ (d-c)n - \frac{c(d-2d')}{3} \right\}. \end{aligned}$$

The period of this quasipolynomial is d . The relative volume of $[0, 1]^4 \cap \mathcal{H}_{12}^{d/c}$ is

$$\bar{\alpha}^{\pm d/c} = \frac{3d-c}{3d^2}.$$

The number of ordered triples of positions that attack each other along a single line of slope d/c is

$$(4.5) \quad \begin{aligned} \alpha_3^{d/c}(n) &:= E_{(0,1)^6 \cap \mathcal{H}_{12}^{d/c} \cap \mathcal{H}_{23}^{d/c}}(n+1) \\ &= \left\{ \left(\frac{1}{d^2} - \frac{c}{2d^3} \right) n^4 + \frac{c}{2d} n^2 \right\} \\ &\quad + \frac{d'(d-d')}{d^3} \left\{ 3(d-c)n^2 - (d-2c)(d-2d')n + \frac{3cd(d-d')}{2} \right\}. \end{aligned}$$

The period of this quasipolynomial is d . The relative volume of $[0, 1]^6 \cap \mathcal{H}_{12}^{d/c} \cap \mathcal{H}_{23}^{d/c}$ is $(2d-c)/2d^3$.

Each quasipolynomial has an invariant part (in the first set of braces), which is independent of d' , i.e., of the residue class of n , and a periodic part (in the second set of braces), which depends on d' . When n is a multiple of d , then $d' = 0$ and the equations simplify to the invariant part.

If the degree is e (that is 3 for $\alpha^{d/c}$ and 4 for $\alpha_3^{d/c}$), the coefficients $\gamma'_{e-i}(d')$ of the periodic part have the alternating symmetry $\gamma'_{e-i}(d-d') = (-1)^i \gamma'_{e-i}(d')$. For instance, in Equation (4.4) $e = 3$ and the periodic part of the coefficient of n (so $i = 2$) is $\frac{d'(d-d')}{d^2}(d-c)$, which is invariant under the mapping $d' \mapsto d-d'$ (for $1 \leq d' \leq d$). That is, for any $k \in \mathbb{Z}_{>0}$ and any $d' = 1, 2, \dots, d-1$, $\gamma_1(kd+d') = \gamma_1(kd+(d-d'))$.

The fact that there is no second leading term will be important in examples.

Note that $\mathcal{H}_{12}^{d/c} \cap \mathcal{H}_{23}^{d/c} = \mathcal{H}_{12}^{d/c} \cap \mathcal{H}_{13}^{d/c} \cap \mathcal{H}_{23}^{d/c}$; any two of these three hyperplanes have the same intersection as all three.

Proof. The number of attacking pairs is the sum over all lines with slope d/c of $|l_B^{d/c}(b)|^2$. From Lemma 4.2 we can write out the total number:

$$\alpha^{d/c}(n) = 2cd \sum_{l=1}^{\delta-1} l^2 + [cd + cd' + (d-d')(n-c\delta)]\delta^2 + [d'(n-c\delta)](\delta+1)^2,$$

which simplifies to Equation (4.4) after eliminating δ via $\delta = (n-d')/d$.

If $c < d$ the period d follows from examining the coefficient of n , which equals $c/3$ only when $d' = 0$. If $c = d$, then both equal 1 and the period is $d = 1$.

The computation for attacking triples is similar. The total number of such triples is

$$\alpha_3^{d/c}(n) = 2cd \sum_{l=1}^{\delta-1} l^3 + [cd + cd' + (d-d')(n-c\delta)]\delta^3 + [d'(n-c\delta)](\delta+1)^3,$$

which simplifies to Equation (4.5). The constant term has period exactly d . \square

For the piece \mathbb{P} we have the formula

$$\alpha_{\mathbb{P}}(n) = \sum_{(c,d) \in \mathbf{M}} \alpha^{d/c}(n) - (|\mathbf{M}| - 1)n^2,$$

which is the sum over all moves $(c, d) \in \mathbf{M}$ of the number of placements of two labelled pieces that attack along that direction, reduced by the overcount of two pieces on the same square, which should be counted only once per square. By Equation (4.1),

$$(4.6) \quad u_{\mathbb{P}}(2; n) = \frac{1}{2!} o_{\mathbb{P}}(2; n) = \frac{1}{2} \left[n^4 - \sum_{(c,d) \in \mathbf{M}} \alpha^{d/c}(n) + (|\mathbf{M}| - 1)n^2 \right].$$

the number of placements of two labelled pieces that do not attack each other. Then by Proposition 4.3 we have an explicit formula for $u_{\mathbb{P}}(2; n)$. For relatively prime integers c and d , define $c^* := \min(|c|, |d|)$ and $d^* := \max(|c|, |d|)$ and also $d' := n \bmod d^*$.

Corollary 4.4.

$$\begin{aligned} u_{\mathbb{P}}(2; n) &= \frac{1}{2!} o_{\mathbb{P}}(2; n) \\ &= \frac{1}{2} \left[n^4 - \sum_{(c_r, d_r) \in \mathbf{M}} \left(\frac{1}{d_r^*} - \frac{c_r^*}{3(d_r^*)^2} \right) n^3 + (|\mathbf{M}| - 1)n^2 - \sum_{(c_r^*, d_r^*) \in \mathbf{M}} \frac{c_r^*}{3} n \right. \\ &\quad \left. - \sum_{(c_r, d_r) \in \mathbf{M}} \frac{d'_r(d_r^* - d'_r)}{(d_r^*)^2} (d_r^* - c_r^*) n + \sum_{(c_r, d_r) \in \mathbf{M}} \frac{c_r^*(d_r^* - 2d'_r)}{3} \right]. \end{aligned}$$

Again, notice that the three highest terms are independent of the residue class of n .

4.3. Near-leading coefficients.

The vertices of $\mathcal{P} = [0, 1]^q$ are integers; thus, Lemma 2.3 implies that γ_{2q-1} is a constant. We can evaluate it exactly and prove that γ_{2q-2} is also constant. We shall see in later sections that, depending on the piece, several more coefficients may be constant.

Theorem 4.5. *On the square board, regardless of the particular piece \mathbb{P} chosen, the coefficients γ_{2q-1} and γ_{2q-2} are constant. The value of γ_{2q-1} is*

$$(4.7) \quad \gamma_{2q-1} = -\frac{1}{2(q-2)!} \sum_{(c,d) \in \mathbf{M}} \left(\frac{1}{\max(|c|, |d|)} - \frac{\min(|c|, |d|)}{3 \max(|c|, |d|)^2} \right).$$

Proof. In Equation (2.1) the terms of degree $2q - 1$ come from $E_{\mathcal{P}}^{\circ}(t)$ and from the leading terms of $E_{\mathcal{H} \cap \mathcal{P}^{\circ}}(t)$ for hyperplanes $\mathcal{H} \in \mathcal{L}(\mathcal{P}^{\circ}, \mathcal{A}_{\mathbb{P}})$. The former equals n^{2q} so it contributes nothing to lower terms. The latter contributes the volumes of $\mathcal{H} \cap \mathcal{P}$. Hence, Equation (2.1) gives the formula

$$\gamma_{2q-1} = -\frac{1}{q!} \sum_{\mathcal{H} \in \mathcal{A}_{\mathbb{P}}} \text{vol}(\mathcal{H} \cap [0, 1]^{2q}).$$

Write $\mathcal{H}_{ij}^{d/c}$ for the hyperplane $c(y_j - y_i) = d(x_j - x_i)$ in \mathbb{R}^{2q} , so

$$\mathcal{A}_{\mathbb{P}} = \{\mathcal{H}_{ij}^{d/c} : (c, d) \in \mathbf{M}, 1 \leq i < j \leq q\}.$$

We may assume $0 \leq c \leq d$ as the contributions of all of $\mathcal{H}_{ij}^{\pm d/c}$ and $\mathcal{H}_{ij}^{\pm c/d}$ are the same. The term of $\mathcal{H}_{ij}^{d/c}$ in the Möbius expansion (2.1) is

$$(4.8) \quad \mu(\hat{0}, \mathcal{H}_{ij}^{d/c}) E_{\mathcal{H}_{ij}^{d/c} \cap [0,1]^{2q}}(n+1) = -\alpha_{\mathbb{P}}(n) n^{2q-4}.$$

Then $\bar{\alpha}^{d/c} = \text{vol}(\mathcal{H}_{ij}^{d/c} \cap [0,1]^{2q})$, so

$$(4.9) \quad \gamma_{2q-1} = \frac{1}{q!} \sum_{1 \leq i < j \leq q} \sum_{(c,d) \in \mathbf{M}} \text{vol}(\mathcal{H}_{ij}^{d/c} \cap [0,1]^{2q}) = \frac{1}{q!} \binom{q}{2} \sum_{(c,d) \in \mathbf{M}} \bar{\alpha}^{d/c}.$$

The volume $\bar{\alpha}^{d/c}$ is given by Proposition 4.3.

The Möbius function $\mu(\hat{0}, \mathcal{H}) = -1$; that accounts for the sign.

The coefficient γ_{2q-2} is determined by subspaces of codimension 2 or less. The contribution from codimension 2 is independent of n since it is the sum of leading coefficients. By Proposition 4.3, subspaces of codimension 1 provide a contribution of zero. The contribution from $\mathcal{P}^\circ = (0,1)^{2q}$ is zero. Thus, γ_{2q-2} is independent of n . \square

4.4. One-move riders.

Even when a piece has only one attacking move, the fourth quasipolynomial coefficient γ_{2q-3} may be non-constant with a large period. We prove that here (without appealing to the general theory). This leads us to propose that periodic variability of higher quasipolynomial coefficients occurs, not due to of the number of attacking moves, but because of their slopes.

Consider a piece \mathbb{P} with move set $\mathbf{M} = \{(c,d)\}$, where c and d are relatively prime integers such that $0 < c \leq d$. By symmetry, this covers all types of move except $(1,0)$ and $(0,1)$, which are elementary.

Proposition 4.6. *For a piece \mathbb{P} with move set $\mathbf{M} = \{(c,d)\}$ where $d \neq 0$,*

$$u_{\mathbb{P}}(1; n) = n^2.$$

$$u_{\mathbb{P}}(2; n) = \frac{1}{2} \left[n^4 - \left\{ \frac{1}{d} - \frac{c}{3d^2} \right\} n^3 - \left\{ \frac{c}{3} + \frac{d'(d-d')(d-c)}{d^2} \right\} n + \frac{cd'(d-d')(d-2d')}{3d^2} \right].$$

$$\begin{aligned} u_{\mathbb{P}}(3; n) &= \binom{n^2}{3} - \sum_{l \in \mathbf{L}(n)} \binom{l}{3} - \sum_{l \in \mathbf{L}(n)} \binom{l}{2} [n^2 - l] \\ &= \binom{n^2}{3} + 2 \sum_{l \in \mathbf{L}(n)} \binom{l}{3} - (n^2 - 2) \sum_{l \in \mathbf{L}(n)} \binom{l}{2}, \end{aligned}$$

$$\begin{aligned} u_{\mathbb{P}}(4; n) &= \binom{n^2}{4} - \sum_{l \in \mathbf{L}(n)} \binom{l}{4} - \sum_{l \in \mathbf{L}(n)} \binom{l}{3} [n^2 - l] - \sum_{\{l, l'\} \subset \mathbf{L}(n)} \binom{l}{2} \binom{l'}{2} \\ &\quad - \sum_{l \in \mathbf{L}(n)} \binom{l}{2} \left[\binom{n^2 - l}{2} - \sum_{l' \in \mathbf{L}(n)} \binom{l'}{2} + \binom{l}{2} \right], \end{aligned}$$

where $\mathbf{L}(n) := \mathbf{L}^{d/c}(n)$.

Proof. The formula for $u_{\mathbb{P}}(1; n)$ is trivial. Equations (4.1) and (4.4) imply the value of $u_{\mathbb{P}}(2; n)$; a combinatorial count similar to that for $u_{\mathbb{P}}(3; n)$ and $u_{\mathbb{P}}(4; n)$ gives the same result.

The reasoning for $q = 3$ and 4 is by direct combinatorial arguments. For instance, $u_{\mathbb{P}}(4; n)$ is the number of placements of four non-attacking pieces, which we count by placing four

pieces on any of the n^2 positions on the board and removing those where at least two pieces attack. We must remove the cases where four pieces are in the same line $l^{d/c}(b)$, those where three pieces are in the same line and the fourth is in another line, those where two pieces are in the same line $l^{d/c}(b)$ and the remaining two are both in another line $l^{d/c}(b')$, and last, those where two pieces are attacking and the remaining two pieces attack none of the others. \square

The equation for $u_{\mathbb{P}}(3; n)$ agrees with the formulas for partial queens \mathbb{Q}^{10} and \mathbb{Q}^{01} in Part II. The formula for $u_{\mathbb{P}}(q; n)$ where $q \geq 5$ seems impractical to solve combinatorially.

The example with $(c, d) = (1, 2)$ simplifies nicely when $q = 2, 3$.

Proposition 4.7. *For a piece \mathbb{P} with move set $\mathbf{M} = \{(1, 2)\}$, the following formulas hold.*

$$u_{\mathbb{P}}(2; n) = \left\{ \frac{n^4}{2} - \frac{5n^3}{24} - \frac{11n}{48} \right\} + (-1)^n \frac{n}{16},$$

$$u_{\mathbb{P}}(3; n) = \left\{ \frac{n^6}{6} - \frac{5n^5}{24} + \frac{n^4}{16} - \frac{11n^3}{48} + \frac{7n^2}{48} + \frac{1}{32} \right\} + (-1)^n \left\{ \frac{n^3}{16} - \frac{n^2}{16} - \frac{1}{32} \right\}.$$

Proof. The value of $u_{\mathbb{P}}(2; n)$ is simplified from Proposition 4.6. That of $u_{\mathbb{P}}(3; n)$ is obtained by simplifying the formula

$$u_{\mathbb{P}}(3; n) = \begin{cases} \binom{n^2}{3} + 8\binom{n/2}{4} + 2(n+2)\binom{n/2}{3} \\ \quad - (n^2 - 2)[4\binom{n/2}{3} + (n+2)\binom{n/2}{2}] & \text{for } n \text{ even,} \\ \binom{n^2}{3} + 8\binom{(n+1)/2}{4} + (n-1)\binom{(n-1)/2}{3} + (n+1)\binom{(n+1)/2}{3} \\ \quad - (n^2 - 2)[4\binom{(n+1)/2}{3} + \frac{n-1}{2}\binom{(n-1)/2}{2} + \frac{n+1}{2}\binom{(n+1)/2}{2}] & \text{for } n \text{ odd,} \end{cases}$$

also from Proposition 4.6. \square

In each formula the fourth coefficient has period 2. This and Lemma 4.2 suggest a generalization.

Conjecture 4.8. *For a one-move rider with basic move $(|c|, |d|)$, the period of γ_{2q-3} is $\max(c, d)$.*

This and the bishop, queen, and nightrider suggest a greater generalization.

Conjecture 4.9. *For any rider, the period of γ_{2q-3} is the least common multiple of the numbers $\max(|c|, |d|)$ for all basic moves $(c, d) \in \mathbf{M}$.*

5. SPECIAL PIECES: A PREVIEW

Our ultimate objective is to produce exact formulas for particular pieces. For instance, Kotěšovec proposes formulas for the coefficients γ_{2q-1} and γ_{2q-2} of the counting quasipolynomials for queens and bishops and other riders [9, third ed., pp. 13, 210, 223, 249, 652, 663]. In Part II we prove several results that make Kotěšovec's formulas rigorous, including a generalization of those conjectures. Here we give a foretaste of the results.

Many of the properties of the rook, bishop, and queen generalize to pieces whose moves are a subset of those of the queen. We call such pieces *partial queens* and examine them closely in Part II. The partial queen \mathbb{Q}^{hk} has h horizontal and vertical moves and k 45°-diagonal moves (where $h, k \leq 2$); thus, \mathbb{Q}^{20} is the rook, \mathbb{Q}^{02} is the bishop, and \mathbb{Q}^{22} is the queen.

5.1. The rook.

For rooks \mathbb{R} the formula is $u_{\mathbb{R}}(q; n) = q! \binom{n}{q}^2$ is known to everyone. The number of unlabelled combinatorial configuration types of q rooks is $q!$.

5.2. The bishop.

Table 5.1 shows descriptive data for nonattacking placements of few bishops.

	Types	Period	Denom	lcmd
$q = 1$	1	1	1	1
2	2	1	1	2
3	6	2	2	4
4	24	2	2	8
5	120	2	2	16
6	720	2	2	32

TABLE 5.1. The number of (unlabelled) combinatorial types of configurations of q (unlabelled) bishops in an $n \times n$ square board, the period of the counting quasipolynomial, the denominator of the inside-out polytope, and (“lcmd”) the determinantal upper bound on the period.

The quasipolynomial formulas for up to 6 bishops were published by Kotěšovec in [9]. Most of them were found by him using his heuristic procedure: he calculates the values $u_{\mathbb{B}}(q; n)$ for many values of n , looks for an empirical recurrence relation, infers from it the generating function, and from that deduces the quasipolynomial. That approach, while excellent for finding formulas, does not prove their validity because it does not bound the possible period. In Part II we provide the necessary upper bound of 2, which rigorously establishes the period and the correctness of Kotěšovec’s quasipolynomial formulas. The proof relies on signed graph theory applied to the bishops hyperplane arrangement $\mathcal{A}_{\mathbb{B}}$.

Kotěšovec conjectured formulas for γ_{2q-1} , γ_{2q-2} , and γ_{2q-3} , which we prove in Part II since the bishop is the partial queen \mathbb{Q}^{02} . The formulas imply constancy of these coefficients. We also prove that γ_{2q-4} is constant and we explain why γ_{2q-6} should be non-constant. We believe γ_{2q-5} can be proved constant by the method used to establish the bishops period, but with more difficulty.

As for the determinantal upper bound on the period (from Section 3.2.2), for the move matrix for bishops we have $\text{lcmd}(M_{\mathbb{B}}) = 2$. It follows that the period of $u_{\mathbb{B}}(q; n)$ divides 2^{q-1} . As the true value is 1 or 2 for each q , this example makes plain that the determinantal bound is very weak.

The number of combinatorial configuration types of q bishops is $u_{\mathbb{B}}(q; -1) = q!$, the same as for rooks.

5.3. The queen.

Table 5.2 gives data on configuration types, etc. Unlike in the case of bishops, the period of $u_{\mathbb{Q}}(q; n)$ is not simple and we have no general formula.

For two queens Equation (4.1) gives the previously known formula

$$u_{\mathbb{Q}}(2; n) = \frac{n^4}{2} - \frac{5n^3}{3} + \frac{3n^2}{2} - \frac{n}{3}.$$

	Types	Period	Denom	lcmd
$q = 1$	1	1	1	1
2	4	1	1	2
3	36	2	2	4
4	574*	6*	6	24
5	14206*	60*	—	—
6	501552 [†]	840 [†]	—	—
7	—	360360 [‡]	—	—

TABLE 5.2. The number of combinatorial configuration types of q (unlabelled) queens in an $n \times n$ square board, with the period, the denominator, and the determinantal bound. Periods without denominators are unproved.

* is a number deduced from a formula in [9].

[†] are deduced from the formula of Karavaev ([7] and [12, Sequence A176186]).

[‡] is deduced from the generating function in [12, Sequence A178721].

The number of combinatorial types of configuration of two (unlabelled) queens is $u_{\mathbb{Q}}(2; -1) = 4$, as one would expect from the geometry of two queens on the board. The period of 1 properly divides the determinantal upper bound of 2.

The quasipolynomial formula for three queens follows from our treatment of partial queens in Part II:

$$u_{\mathbb{Q}}(3; n) = \left\{ \frac{n^6}{6} - \frac{5n^5}{3} + \frac{79n^4}{12} - \frac{25n^3}{2} + 11n^2 - \frac{43n}{12} + \frac{1}{8} \right\} + (-1)^n \left\{ \frac{n}{4} - \frac{1}{8} \right\}.$$

Kotěšovec conjectured general formulas for γ_{2q-1} and γ_{2q-2} for any number of queens, based on the previously known and (by him) heuristically derived formulas for $u_{\mathbb{Q}}(q; n)$ for small q . In Part II we prove his conjectures along with the formula for γ_{2q-3} , and constancy of γ_{2q-4} ; we also have reason to expect γ_{2q-5} to vary.

5.4. The nightrider.

For nightriders \mathbb{N} , by subtracting the number of attacking pairs of squares in all knight-like diagonals from the total number of pairs, we find that

$$u_{\mathbb{N}}(2; n) = \left\{ \frac{n^4}{2} - \frac{5n^3}{6} + \frac{3n^2}{2} - \frac{11n}{12} \right\} + (-1)^n \frac{n}{4}.$$

This formula was found independently by Kotěšovec. Kotěšovec [9] has an enormous formula for three nightriders (undoubtedly correct, though unproved) that is too complicated to reproduce here.

A direct consequence of Theorem 4.5 is that we know the second coefficient of the counting quasipolynomial, $\gamma_{2q-1} = -\frac{5}{6(q-2)!}$; this formula was conjectured by Kotěšovec. Further results and some new conjectures appear in Part II.

The move matrix has $\text{lcmd}(M_{\mathbb{N}}) = 60$ [6, Example 3].

	Types	Period	Denom	lcmd
$q = 1$	1	1	1	1
2	7	2	2	60
3	36^*	60^*	60	3600
4	—	—	14559745200	14290972303608000

TABLE 5.3. The number of combinatorial configuration types of nonattacking placements of q (unlabelled) nightriders in an $n \times n$ square board; also, the period, denominator, and determinantal bound.

6. QUESTIONS, EXTENSIONS

Work on nonattacking chess placements raises many questions, several of which have general interest. Besides Conjectures 4.8 and 4.9, and others to appear in Part II, we propose the following questions and directions.

6.1. Riders versus non-riders.

Kotěšovec's many formulas are quasipolynomials only for riders. For all others he gets an eventual polynomial, as in our analysis of pieces on a $k \times n$ board where k is fixed [4]. It seems clear that the reason he does not get a quasipolynomial is that, with nonriders, not all moves have unbounded distance, so Ehrhart theory does not apply. The reason he gets an eventual polynomial is less apparent. We believe it is, in essence, that the count is the number of ways to place a finite number of “tight” nonattacking configurations involving a total of q pieces so that no two tight configurations overlap, each tight configuration that can fit on the board contributes a polynomial to the total count, and for large n the board is big enough that every possible tight configuration can fit. How to make this intuitive statement precise is not precisely clear.

6.2. Varied moves.

Our counting method extends to a much more general situation. For convenience we assume distinguishable pieces, $\mathbb{P}_1, \dots, \mathbb{P}_q$. Think of the moves as attacks, and suppose the basic attacks $m_{ij,r}$ may depend on both the attacking piece \mathbb{P}_i and the attacked piece \mathbb{P}_j . This may seem unrealistically general but it permits us to combine more than one interesting type of situation. We form a move matrix M_{ij} from the basic attacks of \mathbb{P}_i on \mathbb{P}_j . Theorem 3.1 and the ensuing discussion of the period remains valid if we take A' (the upper half of the system in Equation (3.1)) to be the matrix in Equation (6.1).

The most realistic case is that where, as in chess, the moves (or attacks) do not depend on the piece being attacked. In that case, $M_{ij} = M_i$, independent of j , and the matrix A' becomes more similar to that of Equation (3.1).

6.3. Higher dimensions.

The inside-out polytope method will apply to boards of higher dimension, such as hypercubical boards $\mathcal{B} = (0, 1)^d$ in particular. However, pieces with multidimensional moves will probably be much more difficult to treat, because for two-dimensional moves m_r , the orthogonal vector m_r^\perp defines the move line so the forbidden configurations in \mathbb{R}^{dq} are determined

$$(6.1) \quad A' = \begin{pmatrix} M_{12} & -M_{12} & 0 & 0 & \cdots & 0 & 0 \\ M_{21} & -M_{21} & 0 & 0 & \cdots & 0 & 0 \\ M_{13} & 0 & -M_{13} & 0 & \cdots & 0 & 0 \\ M_{31} & 0 & -M_{31} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ M_{1q} & 0 & 0 & 0 & \cdots & 0 & -M_{1q} \\ M_{q1} & 0 & 0 & 0 & \cdots & 0 & -M_{q1} \\ 0 & M_{23} & -M_{23} & 0 & \cdots & 0 & 0 \\ 0 & M_{32} & -M_{32} & 0 & \cdots & 0 & 0 \\ 0 & M_{24} & 0 & -M_{24} & \cdots & 0 & 0 \\ 0 & M_{42} & 0 & -M_{42} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & M_{2q} & 0 & 0 & \cdots & 0 & -M_{2q} \\ 0 & M_{q2} & 0 & 0 & \cdots & 0 & -M_{q2} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & M_{q-1,q} & -M_{q-1,q} \\ 0 & 0 & 0 & 0 & \cdots & M_{q,q-1} & -M_{q,q-1} \end{pmatrix}.$$

by a hyperplane; but when $d > 2$ a move line requires more than one equation to define it, so the forbidden configurations are determined by a subspace of codimension $d - 1$.

6.4. A generalization of total dual integrality?

The least common multiple of subdeterminants of the coefficient matrix of the attack hyperplanes (that is, lcmd) turned out to be a very inefficient bound on the period, because it is much larger than the least common denominator of all vertices. This reminds us of the fact that there are totally dual integral matrices which are not totally unimodular; indeed the analogy is close, since total unimodularity means the $\text{lcmd} = 1$. We suggest that a worthy general question about an integral $r \times s$ matrix M is the relationship between $\text{lcmd } M$ and the least common denominator D of all lattice vertices of M , defined as points $\mathbf{z} \in \mathbb{R}^s$ determined by restrictions $A\mathbf{z} \in \mathbb{Z}^r$ where A is any nonsingular matrix consisting of r rows of M . Though D may usually be much less than $\text{lcmd } M$, the cases of equality, being analogs of totally unimodular matrices, might be quite interesting.

DICTIONARY OF NOTATION

a_j, b_j – coeffs of \mathcal{B} boundary ineq (p. 6)	A – grand mtx in Equation (3.1) (p. 6)
b – y -intercept of $l^{d/c}(b)$ (p. 12)	A' – mtx of eqns of forbidden hyps (p. 7)
$(c, d), (c_r, d_r)$ – coords of move vector (p. 5, 12)	B – mtx of coeffs of \mathcal{B} bdry lines (p. 6)
d/c – slope of a line (p. 12)	C – matrix of two moves (p. 10)
d – degree of quasipolynomial (p. 4)	D – denom of (inside-out) polytope (p. 4)
d – dimension of polytope (p. 4)	$E_{\mathcal{P}}^{\circ}$ – open Ehrhart quasipoly (p. 4)
e_j – coefficient of quasipolynomial (p. 4)	$E_{\mathcal{P}, \mathcal{A}}^{\circ}$ – open Ehrhart of inside-out poly (p. 4)
$f(t)$ – quasipolynomial function (p. 4)	I – identity matrix
h – # horiz, vert moves of partial queen (p. 18)	K_q – complete graph (p. 7)
k – # diagonal moves of partial queen (p. 18)	L_{ir} – list in configuration (p. 9)
$l^{d/c}(b)$ – line of slope d/c , y -int b (p. 12)	M – matrix of moves (p. 6)
$l_{\mathcal{B}}^{d/c}(b) = l^{d/c}(b) \cap [n]^2$ (p. 12)	Z – lower left border of $[n]^2$ (p. 12)
$m_r = (c_r, d_r)$, basic move (p. 2)	\mathbf{I}, \mathbf{I}_0 – walk in configuration space (p. 10)
$m_r^{\perp} = (d_r, -c_r)$, orthogonal to m_r (p. 5)	$\mathbf{L}^{d/c}(n)$ – multiset of line sizes (p. 12)
n – size of square board (p. 2)	\mathbf{M} – set of basic moves (p. 2)
$[n] = \{1, \dots, n\}$ (p. 11)	\mathcal{A} – arrangement of hyperplanes (p. 3)
$[n]^2$ – square board (p. 11)	$\mathcal{A}_{\mathbb{P}}$ – move arr of piece \mathbb{P} (p. 5)
$o_{\mathbb{P}}(t)$ – # allowed lab configs (p. 6)	$\mathcal{B}, \mathcal{B}^{\circ}$ – closed, open board polygon (p. 2)
$o_{\mathbb{P}}(q; n)$ – # allowed lab configs (p. 11)	\mathcal{H} – hyperplane (p. 3)
p – period of quasipolynomial (p. 4)	$\mathcal{H}_{ij}^{d/c}$ – hyperplane for move (c, d) (p. 14)
q – # pieces on a board (p. 2)	\mathcal{L} – intersection semilattice (p. 3)
r – move index (p. 2)	\mathcal{P} – polytope (p. 3)
t – dilation (inflation) variable (p. 4)	$(\mathcal{P}, \mathcal{A})$ – inside-out polytope (p. 3)
$u_{\mathbb{P}}(t)$ – # allowed unlab configs (p. 6)	\mathcal{U} – subspace in intersection semilatt (p. 3)
$u_{\mathbb{P}}(q; n)$ – # allowed unlab configs (p. 11)	
$z = (x, y), z_i = (x_i, y_i)$ – piece position (p. 2)	
$\mathbf{z} = (z_1, \dots, z_q)$ – configuration (p. 5)	\mathbb{Q} – rational numbers
	\mathbb{R} – real numbers
$\alpha_{\mathbb{P}}(n)$ – # attacking configurations (p. 12)	\mathbb{R}^{2q} – configuration space (p. 9)
$\alpha^{d/c}(n) = \sum_{b \in \mathbb{R}} l_{\mathcal{B}}^{d/c}(b) ^2$ (p. 14)	\mathbb{Z} – integers
$\bar{\alpha}^{d/c} = \lim_{n \rightarrow \infty} \alpha^{d/c}(n)/n^3$ (p. 14)	
β_j – constant in \mathcal{B} boundary inequality (p. 6)	\mathbb{B} – bishop (p. 19)
γ_{2q-i} – coefficient of $u_{\mathbb{P}}$ (p. 11)	\mathbb{N} – nightrider (p. 20)
δ_{ij} – Kronecker delta (p. 11)	\mathbb{P} – piece (p. 2)
$\delta = \lfloor n/d \rfloor$ (p. 12)	\mathbb{Q} – queen (p. 19)
μ – Möbius function of $\mathcal{L}(\mathcal{P}^{\circ}, \mathcal{A})$ (p. 4)	\mathbb{Q}^{hk} – partial queen (p. 18)
ω – # boundary lines of \mathcal{B} (p. 6)	\mathbb{R} – rook (p. 19)

\mathbf{H}_q (Eta) – incidence matrix of K_q . (p. 7)

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